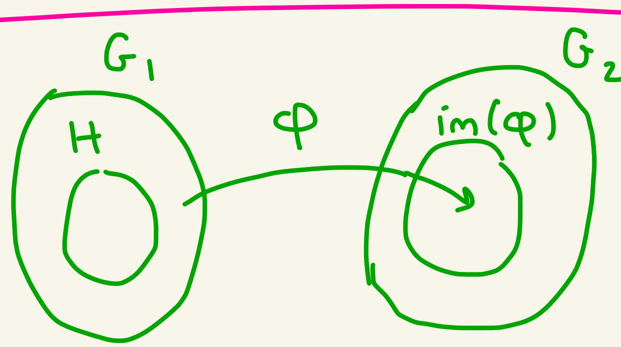
Math 4550

Topic 7 –
First Isomorphism Theorem

Theorem (First Isomorphism Theorem)

Let $\varphi: G_1 \to G_2$ be a homomorphism between two groups G_1 and G_2.

Then,
$$G_1 / \ker(\varphi) \cong \operatorname{im}(\varphi)$$



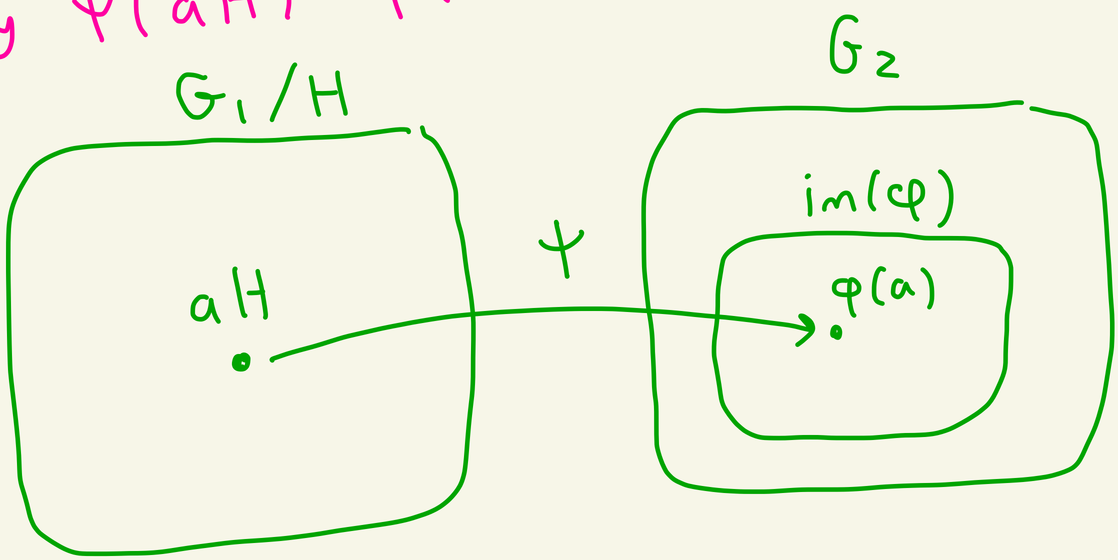
<u>proof:</u> Let $H = \ker(\varphi)$.

Then $H \trianglelefteq G_1$.

So, G_1/H is a group.

Define $\psi: G_1/H \to \operatorname{im}(\varphi)$

by $\psi(aH) = \varphi(a)$.



Claim 1: ψ is well-defined.

pf: Suppose $aH = bH$ for some $a, b \in G_1$.

Then, $b^{-1}a \in H$.

Since $H = \text{Ker}(\varphi)$ we have $\varphi(b^{-1}a) = e_2$ where e_2 is the identity of G_2.

So, $\varphi(b)^{-1}\varphi(a) = e_2$

Thus, $\varphi(a) = \varphi(b)$.

Hence, $\psi(aH) = \varphi(a) = \varphi(b) = \psi(bH)$.

So, ψ is well-defined. $\boxed{\text{claim 1}}$

Claim 2: ψ is an isomorphism

pf: Let $a, b \in G_1$.

Then

$$\psi((aH)(bH)) = \psi(abH) = \varphi(ab) \overset{\text{since } \varphi \text{ is a homomorphism}}{=} \varphi(a)\varphi(b) = \psi(aH)\psi(bH)$$

So, ψ is a homomorphism.

Let's show that ψ is one-to-one.

Suppose $\psi(aH) = \psi(bH)$.

Then $\varphi(a)=\varphi(b)$.

So, $\varphi(b)^{-1}\varphi(a)=e_2$.

Thus, $\varphi(b^{-1}a)=e_2$.

So, $b^{-1}a \in \ker(\varphi)$.

Thus, $b^{-1}a \in H$.

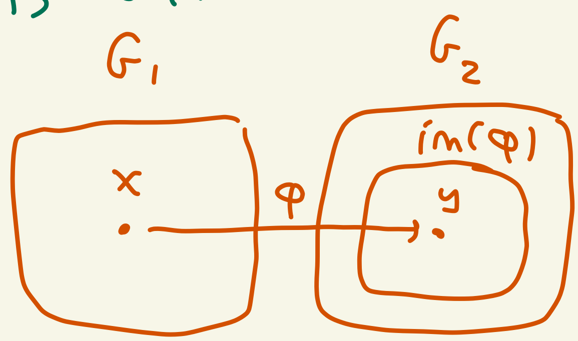
Hence, $aH=bH$.

Therefore ψ is one-to-one.

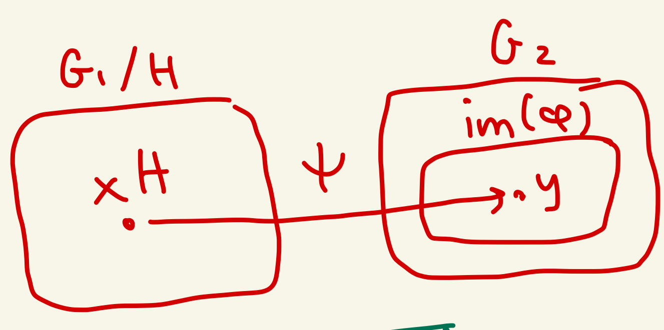
Let's show that ψ is onto.

Let $y \in \operatorname{im}(\varphi)$.

Then there exists $x \in G_1$ with $\varphi(x)=y$.



Hence $xH \in G_1/H$ and $\psi(xH)=\varphi(x)=y$.



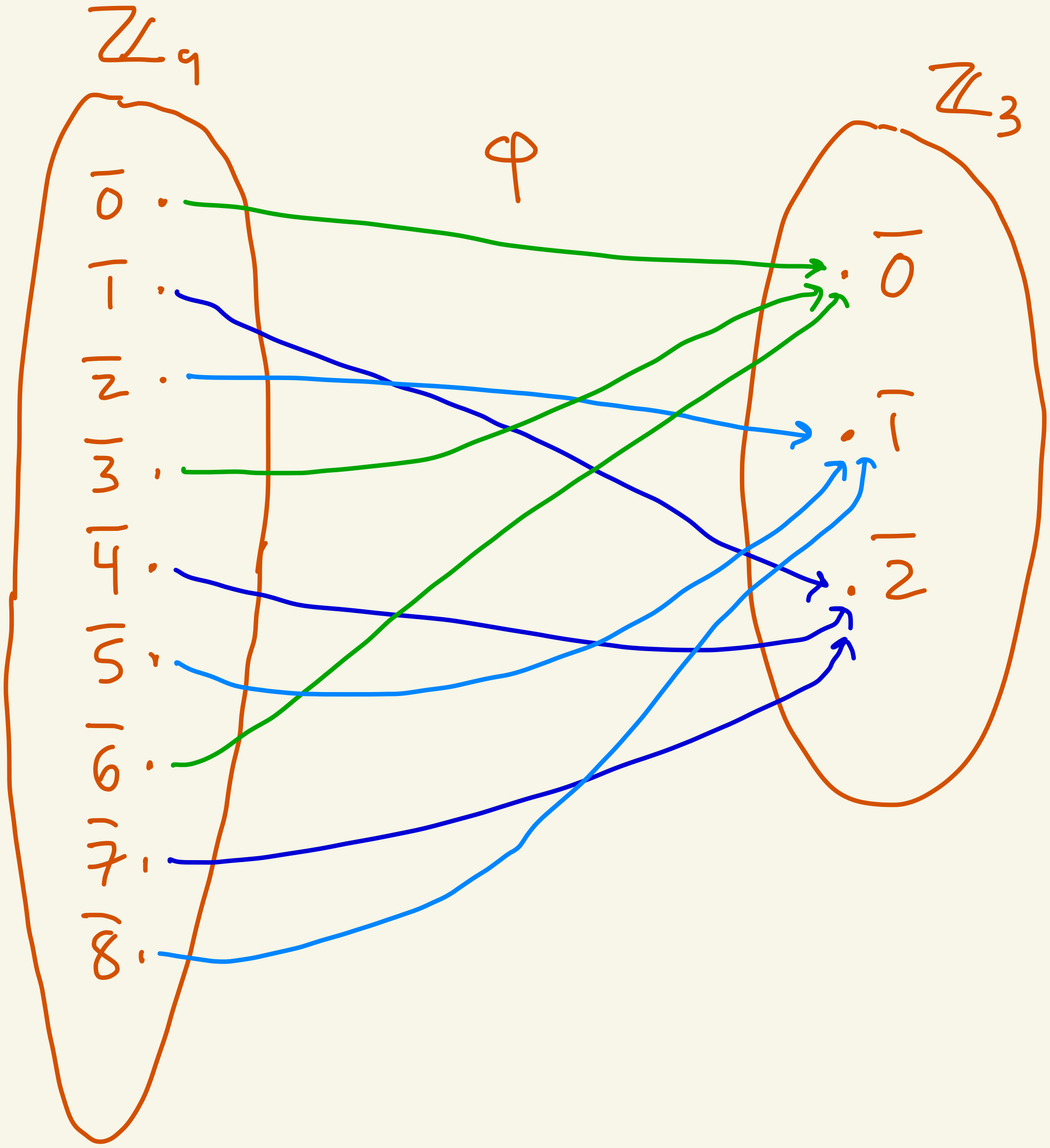
Thus, ψ is onto.

$\boxed{\text{Claim 2}}$

By claim 1 and claim 2 the theorem is proven. $\blacksquare$

Ex: Define $\varphi: \mathbb{Z}_9 \to \mathbb{Z}_3$ where $\underbrace{\varphi(\bar{1}) = \bar{2}}_{\text{defined since } \bar{2} \text{ has order 3 which divides 9.}}$

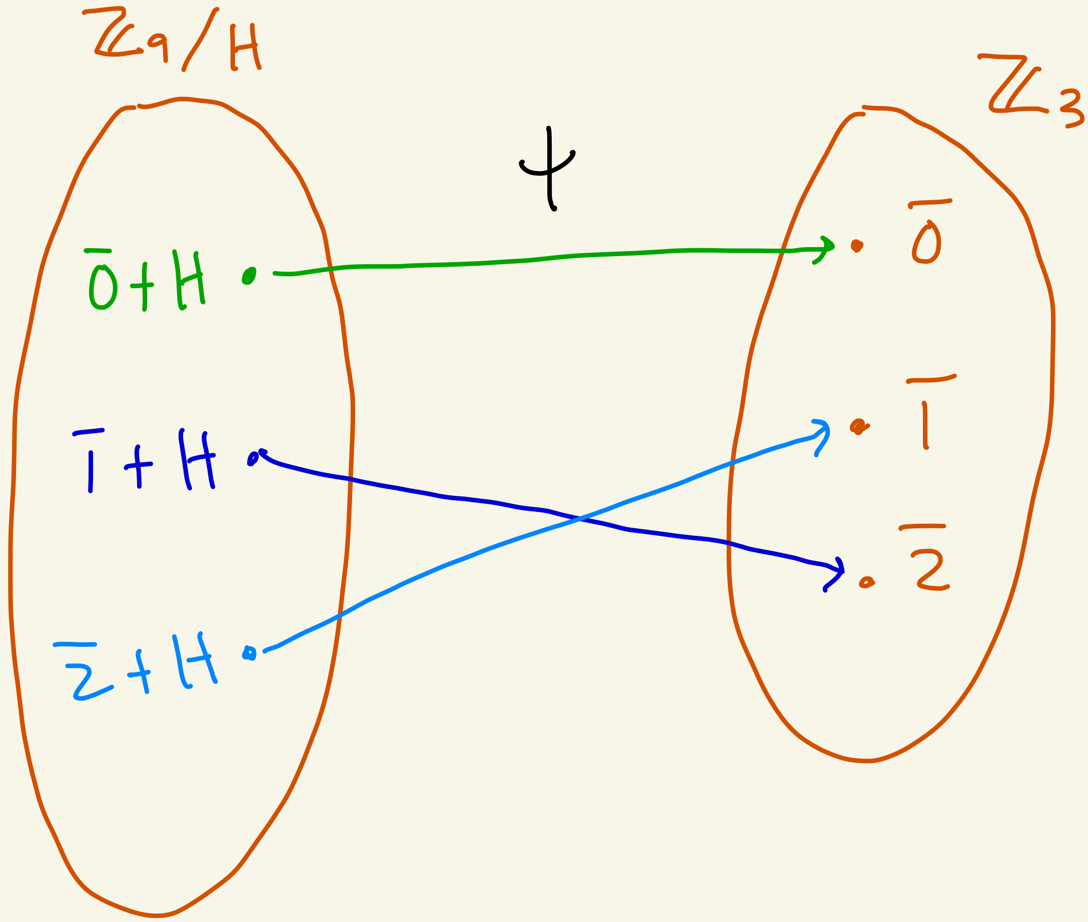
Then we get this:



$H = \ker(\varphi) = \{\bar{0}, \bar{3}, \bar{6}\}$

$\bar{1} + H = \{\bar{1}, \bar{4}, \bar{7}\}$

$\bar{2} + H = \{\bar{2}, \bar{5}, \bar{8}\}$



$$\psi(\bar{0}+H) = \varphi(\bar{0}) = \bar{0}$$
$$\psi(\bar{1}+H) = \varphi(\bar{1}) = \bar{2}$$
$$\psi(\bar{2}+H) = \varphi(\bar{2}) = \bar{1}$$

From the theorem, ψ is an isomorphism.
Thus, $\mathbb{Z}_9/\mathrm{Ker}(\varphi) \cong \mathrm{im}(\varphi) = \mathbb{Z}_3$.

$\blacksquare$

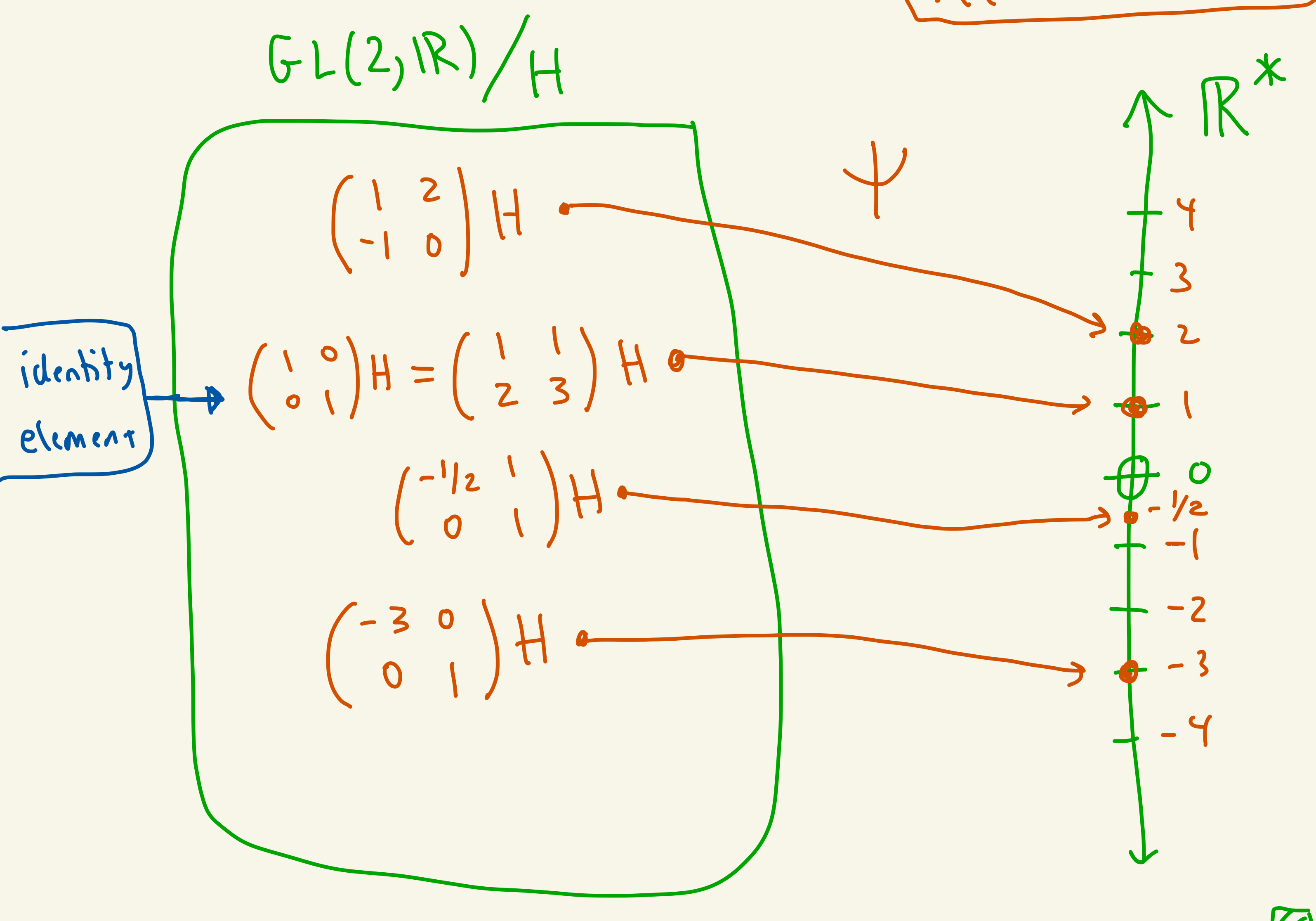
Ex: Let $\varphi : GL(2,\mathbb{R}) \to \mathbb{R}^*$ be defined by $\varphi(A) = \det(A)$.

Recall that φ is a homomorphism with $H = \ker(\varphi) = SL(2,\mathbb{R})$.

Thus, $GL(2,\mathbb{R})/H = GL(2,\mathbb{R})/SL(2,\mathbb{R}) \cong \mathbb{R}^*$

Under the isomorphism $\psi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} H\right) = ad - bc$

$$\boxed{\psi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} H\right) = \varphi\begin{pmatrix} a & b \\ c & d \end{pmatrix}}$$



Ex: Let $\varphi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be given by $\varphi(m,n) = m-n$.

φ is a homomorphism since if $(a,b),(c,d) \in \mathbb{Z} \times \mathbb{Z}$ then

$$\begin{aligned} \varphi((a,b)+(c,d)) &= \varphi(a+c, b+d) \\ &= (a+c)-(b+d) \\ &= (a-b)+(c-d) \\ &= \varphi(a,b) + \varphi(c,d) \end{aligned}$$

φ is onto since given $x \in \mathbb{Z}$ we have that $(x,0) \in \mathbb{Z} \times \mathbb{Z}$ and $\varphi(x,0) = x-0 = x$.

Thus, $\text{im}(\varphi) = \mathbb{Z}$.

By the first homomorphism theorem,

$$\mathbb{Z} \times \mathbb{Z} / \ker(\varphi) \cong \mathbb{Z}.$$

What is $\ker(\varphi)$? We have

$$\begin{aligned} \ker(\varphi) &= \{(a,b) \mid (a,b) \in \mathbb{Z} \times \mathbb{Z} \text{ and } \varphi(a,b) = 0\} \\ &= \{(a,b) \mid (a,b) \in \mathbb{Z} \times \mathbb{Z} \text{ and } a-b = 0\} \\ &= \{(a,b) \mid (a,b) \in \mathbb{Z} \times \mathbb{Z} \text{ and } a = b\} \\ &= \{(x,x) \mid x \in \mathbb{Z}\} \end{aligned}$$

$$= \langle (1,1) \rangle.$$

Let $H = \langle (1,1) \rangle = \ker(\varphi)$.

Then,

$$\mathbb{Z} \times \mathbb{Z} / H \cong \mathbb{Z}.$$

Under the mapping

$$\psi((a,b)+H) = \varphi(a,b) = a-b$$

Example calculation:

$$\begin{array}{ccccc} [(1,2)+H] & + & [(3,-2)+H] & = & (4,0)+H \\ \big\downarrow \psi & & \big\downarrow \psi & & \big\downarrow \psi \\ -1 & + & 5 & = & 4 \end{array}$$